Test 2 Computational Methods of Science/Computational Mechanics, January 2021

Duration: 2 hours and 30 minutes.
In front of the questions, one finds the points. The sum of the points plus 1 gives the end mark for this test. Criteria used for the grading are: demonstration of understanding, logical reasoning, correct use of terminology, correctness of results.

1. Consider the boundary value problem

$$-\frac{\partial}{\partial x}(\bar{u}u) - \frac{\partial}{\partial y}(\bar{v}u) + \frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial x}) - \frac{1}{2}\frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial y}) - \frac{1}{2}\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) = 0,$$

where $\bar{u}(x,y) \equiv 1$, $\bar{v}(x,y) = 1 + x$, and $\mu(x,y) = 1 + xy$. The boundary condtions are: $u(0,y) = 1$, $u(x,0) = \cos(x)$, and $\frac{\partial u}{\partial x}(1,y) = \frac{\partial u}{\partial y}(x,1) = 0$,

   (a) (1 ½ points) Give the finite volume discretization of the field equation on an equidistant grid with mesh size h in both directions.

   (b) (½ point)] Give the discretization of the boundary conditions when the left and bottom boundary conditions are located between the grid points, while the right and top boundary conditions are on the grid points. Also give the positioning of the grid points and the mesh size that results from that choice.

2. Consider the discretization of $\bar{u}\frac{du}{dx} + \mu\frac{d^2u}{dx^2} = 0$, $\frac{du}{dx}(0) = 5$ and $u(1) = 1$ given by

$$\begin{aligned}
&\bar{u}\frac{u_{j+1} - u_j}{h} + \mu\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0 \quad \text{for} \quad j = 2, \cdots, N-1, \\
&\bar{u}\frac{u_2 - u_1}{h} + \mu(\frac{u_2 - u_1}{h^2} - \frac{5}{h}) = 0, \\
&u_N = 1.
\end{aligned}$$

where $x_j = (j - \frac{1}{2})h$, $h = 1/(N - \frac{1}{2})$.

   (a) (1 point) For which combination of $\bar{u}, \mu$ and h will the solution be monotonous? A function is called monotonous on a certain interval if it is either increasing or decreasing on the whole interval.

   (b) (½ point) In which case, there is a restriction on h in order to have a monotonous solution? (If you were not able to give the relation in the previous part you may take it as $\bar{u}h > -4$)

   (c) (1 point) Give the according discretization for the above differential equation and boundary conditions on a non-equidistant grid where the grid points are obtained from a mapping $x = g(\xi)$, where we will use an equidistant grid for ξ. Here g is a monotonically increasing function mapping the interval [0,1] on itself. Make sure that for the choice $g(\xi) = \xi$ the discretization coincides with the one above.

3. Consider the equation

$$\exp(-x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(\cos(x)\frac{\partial u}{\partial x}) \text{ for } x \in (0,1) \text{ and } t > 0$$

with boundary conditions $u(0,t) = 1$ and $u(1,t) = 0$ and initial condition $u(x,0) = 0$.

   (a) (1½ points) Transform the above initial boundary value problem into a system of ODEs $M d\mathbf{c}/dt = A\mathbf{c} + \mathbf{b}$ by a finite element discretization where we have a set of basis functions $\phi_i \in H^1[0,1]$, $i = 1, \cdots, N$ satisfying $\phi_i(0) = \phi_i(1) = 0$.

   (b) (1 point) The difference/Fourier method to get estimates of the relevant eigenvalues for the test equation can only be computed for differences with constant coefficients. Here the coefficients vary. A common way out is to replace the coefficients by constants representing the worst case. In this case, $\exp(-x)$ is replaced by its minimum $1/e$ and $\cos(x)$ by its maximum 1.
   Suppose the basis functions in the previous part are linear and the element size has constant length h. Give the typical rows of matrices M and A if we make the coefficients constant as discussed.

(c) (1 point) Compute the eigenvalues arising from the previous part needed in the test equation.

(d) ($1/2$ point) Show that the eigenvalues of the previous part are in $[-12e/h^2, 0]$.

(e) ($1/2$ point) What is the maximum allowed time step of the Forward Euler method for stability if the eigenvalues are in the interval indicated in the previous part?